

## Combinatorics

1. Many problems in probability theory involve counting the number of ways that an experiment can turn out. In other words, an important aspect of many problems in probability is determining the number of outcomes in the sample space, i.e., determining its **cardinality**. The study of counting techniques for problems of this nature is referred to as **combinatorics**.
2. First, we'll review two basic counting rules that we've already used before. First, the **sum rule** states the following
  - a. If  $A_1, A_2, \dots, A_n$  are disjoint sets, then
$$|A_1 \cup A_2 \cup \dots \cup A_n| = |A_1| + |A_2| + \dots + |A_n|.$$
  - b. In other words, the cardinality of the union of any number of disjoint sets is equal to the sum of the cardinalities of each of those sets individually.
3. Second, the **product rule** states that
  - a. If  $P_1, P_2, \dots, P_n$  are sets (not necessarily disjoint), then
$$|P_1 \times P_2 \times \dots \times P_n| = |P_1| \cdot |P_2| \cdot \dots \cdot |P_n|$$
  - b. In other words, the cardinality of the set of all sequences whose first term is from  $P_1$ , second term is from  $P_2$ , and so on, is equal to the product of the cardinalities of each individual set.
4. Permutations
  - a. A **permutation** of a set  $S$  is a sequence that contains every element of  $S$  exactly once (note that the difference between a set and a sequence is that in a sequence order matters!).
  - b. In counting problems, we will often be interested in the question how many permutations of an  $n$ -element set are there? This question will come up when we need to count all possible orderings of a number of items.
  - c. The total number of permutations of a set  $A$  of  $n$  elements is given by  $n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 1 = n!$
  - d. The reason for this is that there are  $n$  possibilities for choosing the first item in the ordering,  $(n-1)$  for choosing the second, and so on. The symbol "!" is called the **factorial**, and  $n!$  is read "n factorial". By definition,  $0! = 1$ .
  - e. Let  $A$  be an  $n$ -element set, and let  $k$  be an integer between 0 and  $n$ . Then a  **$k$ -permutation** is an ordering of a subset of  $A$  of size  $k$ .
  - f. The total number of  $k$ -permutations of a set  $A$  of  $n$  elements is given by  $P(n, k) = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot (n-k+1) = n!/(n-k)!$
  - g. The reason for this is that there are  $n$  possibilities for the first item in the ordering,  $(n-1)$  for the second, and so on until there are  $(n-k+1)$  possibilities for choosing the final item.
5. Combinations
  - a. With **combinations**, we are interested in counting the total number of subsets of a given size that can be formed from some set  $U$ . For example, how many subsets of 3 elements can be formed from a set  $U$  with 5 distinct elements?

- b. Combinations are often thought of as the number of ways of “choosing” some number of elements, say  $j$  elements, from a set with a total of  $n$  elements. The total number of subsets of size  $j$  that can be chosen from a set of  $n$  elements is denoted  $C(n, j)$  and is pronounced “ $n$  choose  $j$ ”.
  - c. The only difference between permutations and combinations is that with permutations we are counting sequences and with combinations we are counting sets. The only difference between sequences and sets is that in sequences order matters and with sets order does not matter. Thus, the essential difference between permutations and combinations is that with permutations order matters and with combinations order does not matter. So, when we care about order we will use permutations, and when we don't we will use combinations.
  - d. The total number of  $k$ -combinations that can be formed from a set  $A$  with  $n$  elements is given by  **$C(n, k) = n!/((n-k)!k!)$**
  - e. Recall that the number of  $k$ -permutations is given by  $n!/(n-k)!$ , which includes all of the possible ways of ordering each  $k$ -element subset. As we saw before, the number of ways to order  $k$  elements is given by  $k!$ . So, if we divide the number of  $k$ -permutations (which includes all possible orderings) by the number of distinct ways to order each set of  $k$  elements, we will get the number of subsets of size  $k$  where order does not matter, i.e., the number of combinations. In other words, we divide the number of  $k$ -permutations ( $n!/(n-k)!)$  by the number of ways to order each subset of  $k$  elements ( $k!$ ), giving us  $n!/((n-k)!k!)$ , which is the equation given above.
6. Bernoulli Trials
- a. One of the most important applications of combinations in probability is with a specific kind of multi-step experiment called a **Bernoulli trial process**. A Bernoulli trial process is a sequence of  $n$  chance experiments where
    - i. Each experiment has two possible outcomes, which we can call **success** or **failure**.
    - ii. The probability  $p$  of success of each experiment is the same for each experiment. The probability  $q$  of failure is thus  $q = 1 - p$ .
  - b. Examples of Bernoulli trial experiments include,
    - i. A coin is tossed 10 times. The two possible outcomes are heads or tails. The probability of heads on each toss is  $\frac{1}{2}$ .
    - ii. A poll of 1000 people is taken, where they are asked a yes/no question.
  - c. With Bernoulli trial processes, we are often interested in the probability that exactly  $j$  successes will occur in  $n$  trials. For example, in 10 flips of a fair coin, what is the probability that exactly 4 heads come up?
  - d. To calculate a probability of such an event, we first note that the probability of a particular outcome with exactly  $j$  successes equals  $p^j q^{n-j}$ , which corresponds to the value we get by multiplying along the path for such an outcome in a tree diagram.
  - e. To determine the probability of the event where exactly  $j$  successes occur in  $n$  trials, we need to count the total number of outcomes that have this

form. This count is given by  $C(n, j)$ . This count corresponds to picking the  $j$  trials out of  $n$  that will be successes, with the remaining  $n-j$  outcomes being failures.

- f. So, the probability of  $j$  successes in  $n$  trials is given by  $C(n, j) \cdot p^j q^{n-j}$ .